Euler’s Gem or the Birth of Topology

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The Tetrahedron

Count the number $V$ of vertices, $E$ of edges, $F$ of faces:

$V = 4$, $E = 6$, $F = 4$.

Therefore, we have

$V - E + F = 4 - 6 + 4 = 2$. 
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The Cube

Count the number $V$ of vertices, $E$ of edges, $F$ of faces:

$V = 8$, $E = 12$, $F = 6$.

Therefore, we have again $V - E + F = 8 - 12 + 6 = 2$. 
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Theorem (Euler’s formula)

Any convex polyhedron has

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The number \( \chi \) is an invariant for polyhedra and is called the **Euler characteristic**. This formula was discovered around 1750 by Euler.
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Earlier, Descartes (around 1639) discovered a related polyhedral invariant (the total angular defect) but apparently did not notice the above formula itself.
19 proofs

1. Interdigitating Trees
2. Induction on Faces
3. Induction on Vertices
4. Induction on Edges
5. Divide and Conquer
6. Electrical Charge
7. Dual Electrical Charge
8. Sum of Angles
9. Spherical Angles
10. Pick’s Theorem
11. Ear Decomposition
12. ...
(6) Electrical Charge:
This proof is due to William P. Thurston (Fields Medaille 1982). Arrange the polyhedron in space so that no edge is horizontal – in particular, there is exactly one uppermost vertex U and lowermost vertex L.
(6) Electrical Charge:

Put a $+$ charge at each vertex, a $-$ charge at the center of each edge, and a $+$ charge in the middle of each face. Displace all vertex and edge charges into a neighbouring face, then group together all the charges in each face. All charges will cancel out except at L and at U.
(6) Electrical Charge:

The direction of movement is determined by the rule that each charge moves horizontally, counterclockwise as viewed from above. So each face receives the net charge from an open interval along its boundary. This open interval is decomposed into edges and vertices, which alternate. Since the first and last are edges, there is a surplus of one -. So the total charge in each face is zero. All that is left is +2, from L and U.
What is the number \( m \) of Pentagons and \( n \) Hexagons on a football?
The football

What is the number $m$ of Pentagons and $n$ Hexagons on a football?

Each Pentagon is surrounded by 5 Hexagons, and each Hexagon by 3 Pentagons.

$$n = \frac{5m}{3}.$$
The football

The vertices, edges and faces also can be computed in terms of \( m, n \):
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$$V = 5m,$$

$$E = 5m + 6n^2,$$

$$F = m + n.$$
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$$V = 5m, \quad E = \frac{5m + 6n}{2}$$

Therefore, every football has $12$ pentagons and $20$ hexagons.
The football

The vertices, edges and faces also can be computed in terms of $m, n$:

\[ V = 5m, \quad E = \frac{5m + 6n}{2} = \frac{15}{2} m, \]

where we use $n = \frac{5m}{3}$.
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Therefore, every football has 12 pentagons and 20 Hexagons.
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**Figure:** Tetrahedron, Cube, Octahedron, Dodecahedron, Icosahedron
Platonic solids

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Platonic solids

With Euler’s formula we have

\[ 2 = V - E + F = \left( \frac{p}{q} - \frac{p}{2} + 1 \right) F \]
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It can be shown that this equation can only be satisfied for finitely many values of the Schläfli symbol \((p, q)\).
## Platonic solids

<table>
<thead>
<tr>
<th>Polyhedron</th>
<th>Vertices</th>
<th>Edges</th>
<th>Faces</th>
<th>Schläfli symbol</th>
</tr>
</thead>
<tbody>
<tr>
<td>tetrahedron</td>
<td>4</td>
<td>6</td>
<td>4</td>
<td>{3, 3}</td>
</tr>
<tr>
<td>cube</td>
<td>8</td>
<td>12</td>
<td>6</td>
<td>{4, 3}</td>
</tr>
<tr>
<td>octahedron</td>
<td>6</td>
<td>12</td>
<td>8</td>
<td>{3, 4}</td>
</tr>
<tr>
<td>dodecahedron</td>
<td>20</td>
<td>30</td>
<td>12</td>
<td>{5, 3}</td>
</tr>
<tr>
<td>icosahedron</td>
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<td>30</td>
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<td>{3, 5}</td>
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In the 16th century, the German astronomer Johannes Kepler attempted to find a relation between the five known planets at that time (excluding the Earth) and the five Platonic solids.
Generalisations of the Euler characteristic to surfaces, manifolds and topological spaces

Can we define and calculate the Euler characteristic for more general shapes?

Example: the 2-dimensional sphere $S^2$

The Fifa World cup ball from 2006 is a partition of $S^2$ with 6 four sided hourglass-shaped patches and 8 misshapen hexagonal patches, so we have:

$$\chi = V - E + F = 24 - 36 + 14 = 2.$$
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Any other partition or triangulation will give the same $\chi$, so the hole makes a difference!
Generalisations of the Euler characteristic to surfaces, manifolds and topological spaces

Classification of (closed connected orientable) topological surfaces:
Generalisations of the Euler characteristic to surfaces, manifolds and topological spaces

Classification of (closed connected orientable) topological surfaces:

How can we distinguish them?
In which way they are different from the sphere?
Can we classify all of them?
Generalisations of the Euler characteristic to surfaces, manifolds and topological spaces

Figure: $\chi = 0, \quad \chi = -2, \quad \chi = -4$
Generalisations of the Euler characteristic to surfaces, manifolds and topological spaces

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\[ \chi = 2 - 2g = 0, \quad g = 1, \]
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Closed connected orientable topological surfaces are classified by the genus \( g \). Every such surface is topologically equivalent to either the sphere (\( g = 0 \)), the torus (\( g = 1 \)), the double torus (\( g = 2 \)) etc. in other words topologically equivalent to a connected sum of \( g \) tori.
Generalisations of the Euler characteristic to surfaces, manifolds and topological spaces

The polyhedral and topological surfaces we discussed are 2-dimensional finite simplicial complexes. More general the Euler characteristic of an $n$-dimensional simplicial complex is defined as:

$$
\chi = k_0 - k_1 + k_2 - k_3 + k_4 - \cdots ,
$$

where $k_n$ denotes the number of $n$-dimensional simplices. Topological $n$-dimensional manifolds are homotopically equivalent to $n$-dimensional simplicial complexes.
Generalisations of the Euler characteristic to surfaces, manifolds and topological spaces

More generally still, for any topological space $X$, we can define the $n$-th Betti number $b_n$ as the dimension of the vector space $H_n(X, \mathbb{R})$, the $n$-th homology group of $X$. The Euler characteristic is then defined as the alternating sum

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This quantity is well-defined if the Betti numbers are all finite and if they are zero beyond a certain index $n_0$. Betti numbers were studied by H. Poincaré and E. Betti at end of the 19th century and generalised to homology groups by E. Noether in the 1920s. This marked the change from combinatorial topology to algebraic topology.
Generalisations of the Euler characteristic to surfaces, manifolds and topological spaces

Figure: Calabi-Yau manifold

THANK YOU!!!