Tilings and discrete geometry

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Workshop on combinatorial and computational aspects of tilings –London 2008
Tilings by lozenges

We work with lozenge tilings of the plane (tilings with 60° rhombi, dimer covering of the honeycomb graph).
Stepped surfaces

Definition

A stepped surface is defined as a union of faces such that the orthogonal projection onto the diagonal plane $x + y + z = 0$ induces an homeomorphism from the stepped surface onto the diagonal plane.
Lift [Thurston]

Let $T$ be a lozenge tiling of the plane $x + y + z = 0$. Then there exists a unique stepped surface, up to translation by the vector $(1, 1, 1)$, that projects onto $T$ by the orthogonal projection onto the plane $x + y + z = 0$. 
Stepped surface

Definition

A *functional discrete surface* is defined as a union of pointed faces such that the orthogonal projection onto the *diagonal plane* $x + y + z = 0$ induces an *homeomorphism* from the discrete surface onto the diagonal plane.

Being a stepped surface is a *local property*. 
**Stepped surface**

**Definition**

A **functional discrete surface** is defined as a union of pointed faces such that the orthogonal projection onto the **diagonal plane** \( x + y + z = 0 \) induces an **homeomorphism** from the discrete surface onto the diagonal plane.

Being a stepped surface is a **local property**.
A **functional discrete surface** is defined as a union of pointed faces such that the orthogonal projection onto the **diagonal plane** $x + y + z = 0$ induces an **homeomorphism** from the discrete surface onto the diagonal plane.

Being a stepped surface is a **local property**.
Let $v \in \mathbb{R}^3$ and $\mu \in \mathbb{R}$. The **standard arithmetic discrete hyperplane** $\mathcal{P}(v, \mu)$ is defined as

$$\mathcal{P}(v, \mu) = \{x \in \mathbb{Z}^3; \ 0 \leq \langle x, v \rangle + \mu < ||v||_1 \}.$$ 

The **stepped plane** $\mathcal{P}_{v, \mu}$ is defined as the stepped surface whose set of **edges** is $\mathcal{P}(v, \mu)$. 

![Arithmetic discrete planes](image-url)
Some objects...

- **discrete** lines, planes, surfaces,

...and some **transformations/dynamical systems** acting on them

- substitutions

- flips
Generalized substitutions

Generalized substitutions belong to the family of Combinatorial tiling substitutions according to the terminology of [N. Priebe Frank, A primer of substitution tilings of the Euclidean plane].

Motivation

- Define substitution rules acting on stepped surfaces
- Give a geometric version of a multidimensional continued fraction algorithm
Multidimensional substitution

**Exemple [Arnoux-Ito]**

\[ \Theta: 1 \rightarrow \begin{array}{c} 2 \\ 1 \end{array} \quad 2 \rightarrow 3 \quad 3 \rightarrow 1 \]

How to iterate such a rule?


Based on Arnoux-Ito's formalism:

- With a morphism of the free group (+ Hypothesis) \( \sigma \) is associated a generalized substitution \( \Theta(\sigma)^* \).
- We have both local and global information
- Preserves the set of stepped planes and even of stepped surfaces
- We have algebraic properties

\[ \Theta(\sigma \circ \tau)^* = \Theta(\tau)^* \circ \Theta(\sigma)^* \]
Local rules for $\Theta$

\[ 1 \rightarrow \begin{pmatrix} 2 \\ 1 \end{pmatrix} \quad 2 \rightarrow 3 \quad 3 \rightarrow 1 \]
Local rules for $\Theta$

$1 \rightarrow \frac{2}{1}$  $2 \rightarrow 3$  $3 \rightarrow 1$

\[
\begin{array}{c}
\begin{array}{c}
2 \\
1
\end{array} & \rightarrow & \begin{array}{c}
3 \\
2 \\
1
\end{array} \\
3 & 1 & \rightarrow & \begin{array}{c}
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Local rules for $\Theta$

$$1 \rightarrow \frac{2}{1} \quad 2 \rightarrow 3 \quad 3 \rightarrow 1$$

2
1
3
1
$\rightarrow$

3
2
1
$\rightarrow$

2
1
1
$\rightarrow$

2
1
2
$\rightarrow$

2
1
3
$\rightarrow$

2
1
Local rules for $\Theta$

$$1 \rightarrow \frac{2}{1} \hspace{1cm} 2 \rightarrow 3 \hspace{1cm} 3 \rightarrow 1$$
Let $\sigma$ be a substitution on $A$.

Example:

$$\sigma(1) = 12, \ \sigma(2) = 13, \ \sigma(3) = 1.$$ 

The incidence matrix $M_\sigma$ of $\sigma$ is defined by

$$M_\sigma = (|\sigma(j)|_i)_{(i,j) \in A^2},$$

where $|\sigma(j)|_i$ counts the number of occurrences of the letter $i$ in $\sigma(j)$.

### Unimodular substitution

$$\det M_\sigma = \pm 1$$

### Abelianisation

Let $d$ be the cardinality of $A$. Let $l : A^* \to \mathbb{N}^d$ be the abelinisation map

$$l(w) = ^t(\ |w|_1, |w|_2, \cdot \cdot \cdot , |w|_d).$$
Global rule

Let \((x, 1^*), (x, 2^*), (x, 3^*)\) stand for the following faces

Generalized substitution [Arnoux-Ito][Ei]

Let \(\sigma\) be a unimodular morphism of the free group.

\[
\Theta^*_\sigma(x, i^*) = \sum_{k \in A} \sum_{P, \sigma(k) = P \in S} (M_{\sigma^{-1}}^{-1}(x - I(P)), k^*).
\]
Theorem [Arnoux-Ito, Fernique]

Let \( \sigma \) be a unimodular substitution. Let \( \mathbf{v} \in \mathbb{R}^d_+ \) be a nonzero vector. The generalized substitution \( \Theta^*_\sigma \) maps without overlaps the stepped plane \( \mathcal{P}_{\mathbf{v},\mu} \) onto \( \mathcal{P}_{tM\mathbf{v},\mu} \).
**Theorem** [Arnoux-Ito, Fernique]

Let $\sigma$ be a unimodular substitution. Let $v \in \mathbb{R}^d_+$ be a nonzero vector. The generalized substitution $\Theta^*_\sigma$ maps without overlaps the stepped plane $\mathcal{P}_{v,\mu}$ onto $\mathcal{P}_{tM_\sigma v,\mu}$.

Let $\sigma$ be a unimodular **morphism** of the free group. Let $v \in \mathbb{R}^d_+$ be a nonzero vector such that

$$tM_\sigma v \geq 0.$$ 

Then, $\Theta^*_\sigma$ maps without overlaps the stepped plane $\mathcal{P}_{v,\mu}$ onto $\mathcal{P}_{tM_\sigma v,\mu}$. 
Theorem

Let $\sigma$ be a unimodular substitution. The generalized substitution $\Theta^*_\sigma$ acts without overlaps on stepped surfaces.
**A characterization of stepped surfaces by flips**

### Projection

Let \( \pi \) the **orthogonal projection** on the diagonal plane \( x + y + z = 0 \).

### Local finiteness

A sequence of flips \( (\varphi_{s_n})_{n \in \mathbb{N}} \) is said to be **locally finite** if, for any \( n_0 \in \mathbb{N} \), the set \( \{ s_n \in \mathbb{Z}^3, \pi(s_n) = \pi(s_{n_0}) \} \) is bounded.

### Theorem [Arnoux-B.-Fernique-Jamet]

A union of faces \( \mathcal{U} \) is a **stepped surface** if and only if there exist a **stepped plane** \( \mathcal{P} \) and a **locally finite sequence** of flips \( (\varphi_{s_n})_{n \in \mathbb{N}} \) such that

\[
\mathcal{U} = \lim_{n \to \infty} \varphi_{s_n} \circ \ldots \circ \varphi_{s_1}(\mathcal{P}).
\]

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Lozenge tilings  
Substitutions  
Continued fractions
Brun’s algorithm

Brun’s transformation is defined on $[0, 1]^d \setminus \{0\}$ by

$$T(\alpha_1, \ldots, \alpha_d) = \left( \frac{\alpha_1}{\alpha_i}, \ldots, \frac{\alpha_{i-1}}{\alpha_i}, \left\{ \frac{1}{\alpha_i} \right\}, \frac{\alpha_{i+1}}{\alpha_i}, \ldots, \frac{\alpha_d}{\alpha_i} \right),$$

where

$$i = \min\{j | \alpha_j = \|\alpha\|_\infty\}.$$

For $a \in \mathbb{N}$ and $i \in \{1, \ldots, d\}$, we introduce the following $(d + 1) \times (d + 1)$ matrix:

$$B_{a,i} = \begin{pmatrix} a & 1 \\ l_i & 0 \\ 1 & l_{d-i} \end{pmatrix}.$$ 

One has

$$(1, \alpha) = \|\alpha\|_\infty B_{a,i}(1, T(\alpha)).$$

Let $\beta_{a,i}$ be a substitution with incidence matrix $B_{a,i}$, then

$$P_{(1,\alpha),\mu} = \Theta_{\beta_{a,i}}^* (P_{\|\alpha\|_\infty}(1, T(\alpha), \mu)).$$
Brun’s transformation is defined on $[0, 1]^d \setminus \{0\}$ by

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where

$$i = \min\{j \mid \alpha_j = \|\alpha\|_{\infty} \}.$$ 

- Unimodular algorithm
- Weak convergence (convergence of the type $|\alpha - p_n/q_n|$)
- Metric results (natural extension)
Brun expansion of a stepped plane

How to read on the stepped plane

\[ i = \min\{j \mid \alpha_j = \|\alpha\|_\infty \} \] and the partial quotient \( a = [1/\alpha_i]? \)

We thus need entries comparisons and floor computations.

If the above parameters are known, then

\[ \mathcal{P}(1,\alpha),\mu = \Theta_{\beta_{a,i}}^* (\mathcal{P}|\alpha|_\infty(1, T(\alpha)),\mu), \]

where the substitution \( \beta_{a,i} \) has incidence matrix \( B_{a,i} \) with

\[ \|\alpha\|_\infty B_{a,i}(1, T(\alpha)) = (1, \alpha). \]
Brun expansion of a stepped plane $\mathcal{P}_{(1,\alpha)}$

We consider the stepped plane with normal vector $(1, (\alpha_1, \alpha_2))$, with 

$$\alpha = (\alpha_1, \alpha_2) \in [0, 1]^2 \setminus \{0\}.$$ 

One has $\alpha_1 > \alpha_2$.
Brun expansion of a stepped plane

One has \( \left\lfloor \frac{1}{\alpha_1} \right\rfloor = 2. \)
Brun expansion of a stepped plane

Finally, one has $\mathcal{P}(1,\alpha) = \Theta_{\beta_2,1}^*(\mathcal{P}(1,\tau(\alpha)))$.

We thus can define geometrically Brun expansions of a stepped surfaces.
<table>
<thead>
<tr>
<th>Arithmetics</th>
<th>Geometry</th>
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<td>( (1, \alpha_n) \propto B_n(1, \alpha_{n+1}) )</td>
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  with \(t B_n\) incidence matrix of \(\sigma_n\) |
<p>| ((1, \frac{11}{14}, \frac{19}{21}) \propto B_{1,2}(1, \frac{33}{38}, \frac{2}{19})) |</p>
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$(1, \frac{33}{38}, \frac{2}{19}) \propto B_{1,1}(1, \frac{5}{33}, \frac{4}{33})$
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![Diagram of stepped plane](image-url)
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- $\Theta^*_\sigma_n$ represents a substitution rule in the context of continued fractions and tiling theory.

The table summarizes the relationship between arithmetics and geometry in the context of lozenge tilings and substitutions.
Applications

- **Generation** of discrete planes
- **Recognition problem**: Given a set of points in \( \mathbb{Z}^d \), does there exist an arithmetic discrete plane that contains them?
- **Generalized Rauzy fractals** associated with non-algebraic parameters