How do shapes fill space?

Where do we start?

Factsheet 1

Proof (don't be afraid)

The wonderful thing about mathematics is that it deals with simple ideas. For anyone who has sat through a maths class (even us) this sounds like a silly statement. Let me explain. Consider something that most people would say was simple. Like hair colour. Find someone around you and say their hair colour to yourself. Do you think everyone will agree with you? Now try to find someone whose hair might cause debate, perhaps a dark blonde, many people would call brown. Now think of a country like China where there is far less variation in hair colour. Do people notice more subtle differences? Do they notice hair colour at all? You see hair colour is quite a complicated idea when we try to lift it above personal experience. Mathematics is different, we try to consider objects where every feature is defined. We cannot disagree on the meaning of the number "one" and as a result we can prove things. A correct proof is really an argument that cannot be refuted. Proof however can do more, it can take a seemingly complicated situation and make it understandable, or at least less complicated.

Platonic Tilings

Around you you can find some regular shapes. Triangles, squares, pentagons, hexagons...go and have a play with them. Now consider the following question, how many tilings can I make using copies of just one shape? You probably know one already the tiling of squares on your bathroom wall. There are two others. Equilateral triangles and regular hexagons. That is all. How can we be sure of this? We need a proof!

Start with a shape (any regular polygon) and fit tiles round a corner. Take a square for example. Four squares fit exactly round a corner. With a pentagon, however, we have a slight gap after fitting three together. Mathematically we take the angle at a corner. This gives the following:

Shape	Corners	Angle (°)	
		Degrees	Circle fraction
Equilateral triangle	3	60	1/6
Square	4	90	1/4
Regular Pentagon	5	108	3/10
Regular Hexagon	6	120	1/3
Regular Heptagon	7	128.57	5/14
Regular Octagon	8	135	3/8
Regular Nonagon	9	140	7/18
Regular Decagon	10	144	2/5
Regular Enneagon	11	147.27	9/22
Regular Dodecagon	12	150	5/12

We can actually read off from this chart that of these shapes only the equilateral triangle, square and hexagon will fit exactly round a vertex. What about other shapes? Could things suddenly

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work again for a shape with 1531 sides? The answer is no, to do this let us consider the general angle. We have to turn an equal angle at each corner, so at each corner of a shape with n sides we turn 1/n (or 360/n degrees). The angle inside the shape if therefore the angle of a straight line (1/2 or 180 degrees) minus the angle we turned through. We therefore have 1/2-1/n (180-360/n if you want to think in degrees) . This number can only increase but can never get to 1/2. As we have already passed 1/3 for the hexagon no other shape will work.

We are not quite finished. We have shown what cannot work, we now need to show that the three shapes do give us tilings. Can you construct the tilings with 6 equilateral triangles, 4 squares and 3 hexagons round a corner and show they will go on for ever?

Q1 Actually this proof is a bit of a cheat. We left out a key condition, can you find other tilings with these shapes?

Q2 What is the condition we left out?

Generalising

Mathematicians love to take something they understand and push it further. You will see a lot of this on the next posters. We will start gently:

Look at the interior angles listed in the table. There are many ways that we can use different ones to add to 1. For example we can take two pentagons (3/10 x 2 = 3/5) and add a decagon (2/5). Some of these can be used to give tilings, others can't. Some other examples are below.

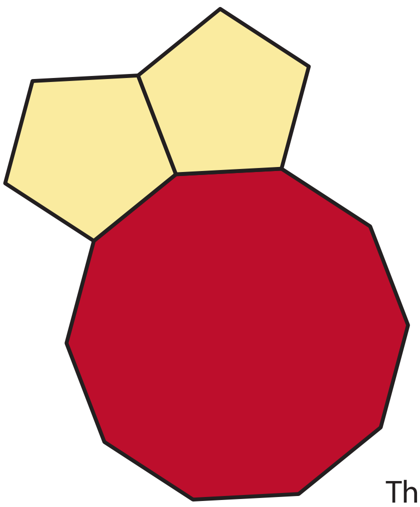
Use the tiles to experiment and answer the following questions, pictures are good here!

Q3 Explain how a square, two triangles and dodecagon can tile the plane.

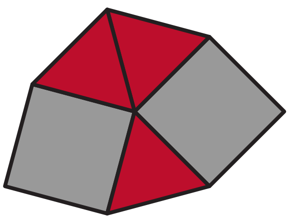
Q4 Explain why a tiling with a decagon and two pentagons around every vertex is not possible.

Q5 Are there any tilings with nonagons and the same tiles round every vertex.

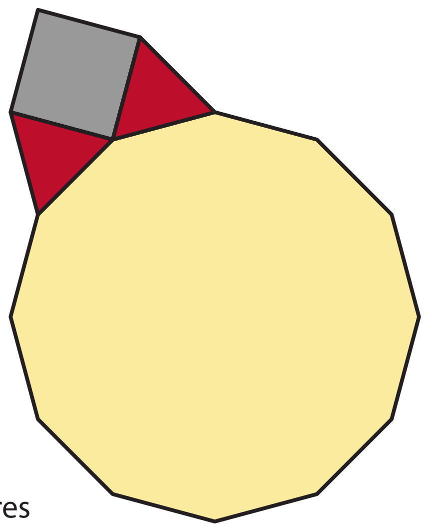
Challenge: Can you find all the tilings with the same configuration at each vertex. (Hint, start by finding all the ways to add the circle fractions add to 1, think about whether the table above is sufficient.)



Two pentagons and a decagon



Three triangles and two squares



Two triangles, a square and a dodecagon

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