SELECTED 2009 THE ROYAL SOCIETY SUMMER SCIENCE EXHIBITION

How do 3d shapes meet?

Factsheet 4

Going round an edge...

We have been looking at placing two dimensional tiles round a corner. This has given tilings on the plane and shapes in three dimensions. What happens when we want to put these together? Think about tetrahedra. Take one and stick a second to its face, now stick on a third, and so on. We are now not going round a corner, we are going round an edge. How many tetrahedra can we place round an egde? We need the angle between two faces. If you are familiar with sine and cosine you can work this out, but to cheat here are the answers for all five regular polyhedra:

Shape	Faces	Edges	Corners	Dihedral angle	
				Degrees	Circle fraction
Tetrahedron	4	6	4	70.53	0.196
Cube	6	12	8	90	0.25
Octahedron	8	12	6	109.47	0.304
Dodecahedron	12	30	20	116.56	0.324
Icosahedron	20	30	12	138.19	0.384

You see from this that five tetrahedra fit round an edge but do not meet up as 0.196 is slightly less than 1/5 = 0.2. Keep this in mind, as you will remember we have a way to deal with this. For the moment we see that the only shape that will meet up is the cube, that meets itself after four go round an edge. Look closer though. the tetrahedron and the octahedron have angles that add to 180, or half a circle (using sine and cosine you can prove that this is exact). In three dimensions therefore we can make a tiling with four cubes round an edge or two octahedra and two tetrahedra.

The fourth dimension

Remember how we got from two dimensions to three dimensions? We took something that did not quite fit when flat (tiles round a point) and folded. We can do the same thing to three dimensional polyhedra that do not quite fill round an edge! The idea of the fourth dimension has entered the popular imagination and has been written about far outside mathematics and science. Unfortunately, much of what has been written is rubbish. So what is the fourth dimension to a mathematician? It is simply a fourth number (coordinate) to describe a point in space. That number could be time, but in many cases, including this exhibit, it is not. Yet this simple space is home to marvels.

The trouble with this space is that we cannot see it visually. In fact we only see three dimensions using some clever tricks. We all combine the two 2-dimensional images from our eyes into a 3d world. The three dimensional world we live in is converted to a two dimensional image using a process called projection. Think about an object casting a shadow in sunlight. Any two points on a straight line with the sun will cast the same shadow. Using a similar method we can show three dimensional shadows of four dimensional objects. The printed images on the poster and here are projections to 2d of 3d models of 4d objects. Even though they are flat we can see the 3d structure thanks to the tricks our eyes perform.

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So from the table we can work out what edge configurations are possible. We can have three four or five tetrahedra round an edge, three or four cubes (with four meeting perfectly) and three octahedra or dodecahedra. Only two icosahedra fit round and edge so these will fold flat. Can you figure out which of these edge configurations corresponds to which polytope on the poster?

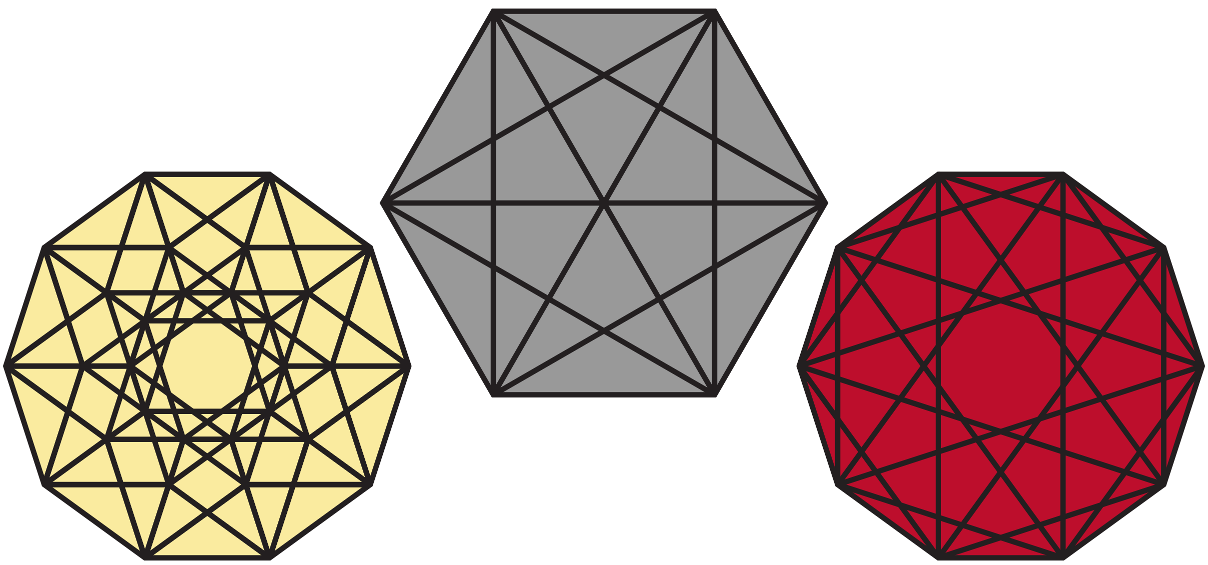
Higher dimensions still

In two dimensions there are an infinite number of regular polygons, in three dimensions there are five regular polyhedra and now in four dimensions we find six regular shapes, called *4d-polytopes*. What about the even harder to visualise fifth dimension, or the 45th? Can we use abstract thinking to explore these higher dimensions where our intuition and visual abilities break down? In this case we can. In order to prove what polyhedra and polytopes could occur we developed a framework. We do not need to be able to play with the shapes to consider the angles, similarly to step from three to four we worked in the abstract. We can conrinue in the abstract. We can now look at the interior angles between three dimensional faces (*cells*) in the 4d polytopes. If you want a real challenge try to work out how! We then get the following table:

Shape	3d Cells	Faces	Edges	Corners	Dihedral angle	
					Degrees	Circle fraction
5-cell	5	10	10	5	70.56	0.196
Tesseract	8	24	32	16	90	1/4
16-cell	16	32	24	8	120	1/3
24-cell	24	96	96	24	120	1/3
120-cell	120	720	1200	600	144	2/5
600-cell	600	1200	720	120	164.52	0.457

We see from this that we can find regular tilings in 4d with Tesseracts, 16-cells and 24-cells. We can now go up to 5d. In a move that hurts your brain if you try to visualise it we now look at shapes going round a 2d face. The only shapes that can have three round a face without meeting are the 5-cell and tesseract, with 3 or 4, 5-cells and 3 -tesseracts possible. This means that there are only 3 regular polytopes in 5 dimensions. In fact this is the case for every higher dimension!

To conclude here are the 2d shadows of the three regular polytopes in 5 dimensions:



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