



# What do you see?

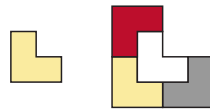
## Factsheet 5

### The Penrose Tiling

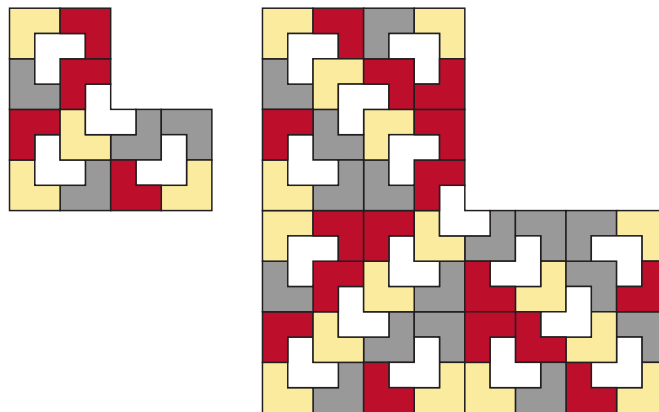
These two shapes have a remarkable property. They can tile the plane but not periodically. In other words the tiling of the plane they produce is not made of a single patch of tiles repeated. We hope here to show you the proof of this fact, remember a proof is in many ways just a simple explanation. To do that we need to do two things. Firstly we need to show that the tiles can in fact tile the plane. Secondly we need to show that no periodic tiling is possible. To show that they tile the plane we need a new construction called a substitution rule.

### Substitution Rule

Think about this L-shaped tile, we can double it in size and then cut this larger shape into four copies of the original:



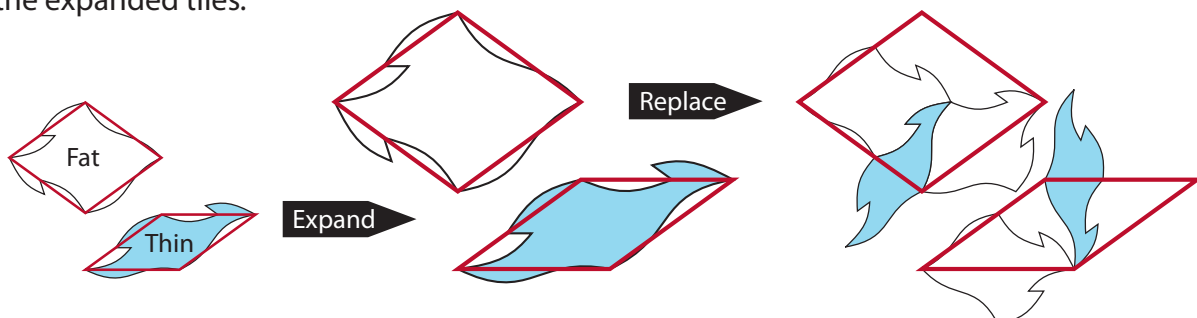
We can now repeat...



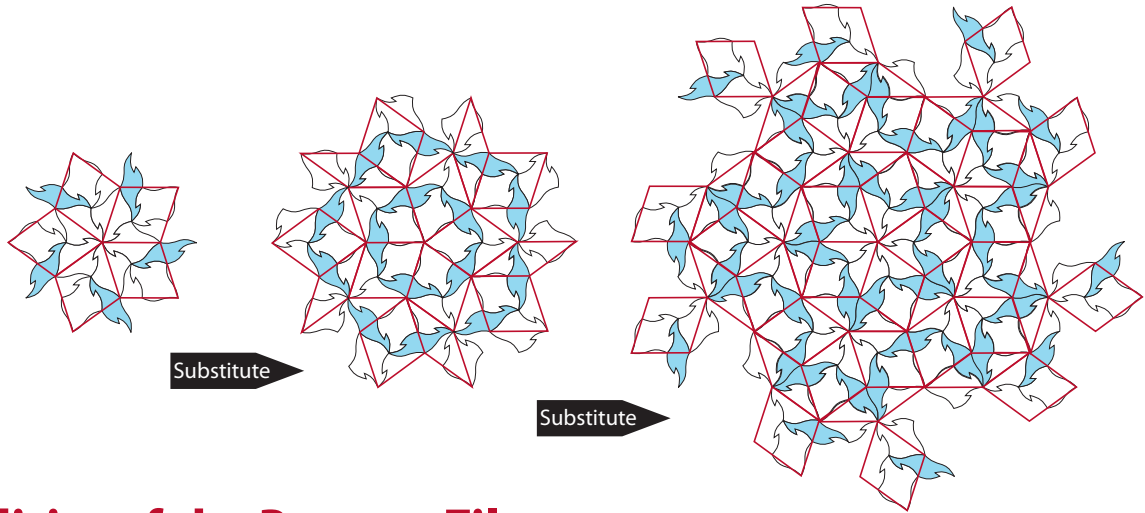
This is a simple example of a substitution rule. In general a substitution rule takes a patch of tiling, expands it and then replaces the larger tiles by patches of the original ones. So in this case we take the L-shaped tile, expand it by a factor of two and then replace it by four copies.

### Penrose Substitution

The substitution rule for the Penrose tiles is a little more complicated. In this case we expand the tiles by the golden ratio:  $(1+\sqrt{5})/2$ , and then replace the larger tiles by the following two patches. Note that these patches of tiles fit together in exactly the same way (though with different edges) as the expanded tiles.



Notice that these two patches can fit together in exactly the same way as the original two tiles. We can therefore think of them as larger versions of the original tiles. We can repeat this to get larger and larger patches of Penrose tiles:



## Aperiodicity of the Penrose Tiles

We now know that the Penrose tiles can fill the plane. Can this tiling be periodic? What happens when we start with a single fat rhomb and start to apply the rule. After one application we have 2 fat rhombs and 1 thin rhomb. At each stage we replace the fat rhombs again by 2 fat and 1 thin, and the thin rhombs by 1 fat and 1 thin. So the numbers of fat and thin rhombs (thin, fat) go as follows:

(0,1), (1,2), (3,5), (8, 13), (21, 34), (55,89), (144, 233)...

This sequence might be familiar to you, these are the Fibonacci numbers, where each number is the sum of the two before. The ratio of a Fibonacci number to its predecessor gets closer and closer to the golden ratio  $(1+\sqrt{5})/2:1$  as the numbers get larger. This is therefore the ratio of fat rhombs to thin rhombs in a tiling generated by the substitution rule (as this involves applying the substitution infinitely often). This is an irrational number, one that cannot be written as a fraction. Think of a periodic tiling: it has one patch repeated, so the ratio of the tiles must be the same as the ratio in the patch. Thus the tiling given by this substitution rule cannot be periodic.

So the Penrose tiles can tile, in a non-periodic way. Is there another way? No. Lets think about how they can fit round a point, some are shown below (we have done a little of the work for you and ruled out a few that cannot continue). Can you show that these patches will either occur in the substitution tiling, or cannot be continued? Try it using the shapes provided.

